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REGULAR ODD RINGS AND NON-PLANAR GRAPHS

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In a previous paper we have announced that a graph is non-planar if and only if it contains a maximal, strict, compact, odd ring. Little has conjectured that the compactness condition may be removed. Chernyak has now published a proof of this conjecture. However, it is difficult to test a ring for maximality. In this paper we show that for odd rings of size five or greater, the condition of maximality may be replaced by a new one called regularity. Regularity is an easier condition to diagnose than is maximality.

1. Introduction

The terminology of this paper generally follows [3] where a new characterisation of planar graphs is announced. However, for convenience and ease of reference, we repeat the relevant definitions here. All graphs are assumed to be finite.

We denote the vertex set of a graph G by VG and its edge set by EG. If G is a directed graph, C is a directed circuit of G, and a and b are distinct vertices of VC, then we use the notation $C(a, b)$ to mean the directed subpath of C with origin a and terminus b. If $a=b$, then $C(a, b)$ means the subpath of C with vertex set $\{a\}$ and empty edge set. Furthermore if P is a path in G with end vertices c and d, then we use IP to denote the set $VP-\{c, d\}$.

If the edges of a graph G can be directed so that every circuit of a set S is a directed circuit, then we say that S is *consistently orientable.* The cyclic sequence $(C_0, C_1, \ldots, C_{n-1})$ of circuits, with $n \geqq 3$, is a *ring* or *n-ring* S in the graph G if

(i) S is consistently orientable,
(ii) $EC_i \cap EC_j \neq \emptyset$ if and only if $i=j$, $i \equiv j+1 \pmod n$ or $i \equiv j-1 \pmod n$, and
(iii) no edge of G belongs to more than two circuits of S.

Let $S=(C_0, C_1, \ldots, C_{n-1})$ be a ring. A $\bar{C}_i C_{i+1}$*-chord,* P, is a subpath of C_{i+1} with end vertices in VC_i such that $(IP \cap VC_i) \cup (EP \cap EC_i) = \emptyset$. $\bar{C}_i C_{i-1}$-chords are defined analogously. (We note that here, and throughout this paper, all subscripts are taken as being modulo n.)

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The ring S is said to be *strict* if $|VC_i \cap VC_j| \leqq 1$ whenever $EC_i \cap EC_j = \emptyset$. Further, S is *maximal* if there does not exist a ring $S' = (C'_0, C'_1, \ldots, C'_{m-1})$ in G such that $\bigcup_{j=0}^{m-1} EC'_j \subseteq \bigcup_{k=0}^{n-1} EC_k$ and $m > n$. We say that S is *odd* if n is odd. Finally, S is *elegant* if for each i there is a unique $\bar{C}_i C_{i+1}$-chord.

Chernyak [1] has proved the conjecture given in [5], namely that a graph is non-planar if and only if it contains a maximal strict odd ring. The usefulness of this characterisation is circumscribed to a certain extent by the fact that the condition of maximality is difficult to test. It would be convenient to obtain a characterisation of non-planar graphs in which this condition is not mentioned. One such characterisation is given in [2] where the following theorem is proved.

Theorem 1.1. *A graph is non-planar if and only if it contains a strict odd elegant ring.* ▮

Further progress can be made by introducing the concept of a regular ring, which we define in the next section. It will be easy to see that each non-planar graph contains a strict regular odd ring. Furthermore we show that any graph which contains a strict regular odd n-ring with $n \geqq 5$ must be nonplanar. Unfortunately we also exhibit a planar graph which contains a regular 3-ring. Thus a graph is non-planar if and only if it contains either a strict regular odd n-ring with $n \geqq 5$ or a maximal 3-ring. (Note that any 3-ring is strict.) This result motivates a study of maximal 3-rings, which may be undertaken elsewhere.

2. Regular rings

A ring $(C_0, C_1, \ldots, C_{n-1})$ is defined to be *regular* if, for each i, every $\bar{C}_i C_{i+1}$-chord contains an edge of EC_{i+2} and every $\bar{C}_{i+1} C_i$-chord contains an edge of EC_{i-1}.

It is easy to see that the graphs $K_{3,3}$ and K_5 each contain a strict regular odd ring. (Such rings are displayed in [4].) It follows by Kuratowski's Theorem that every non-planar graph contains such a ring. Hence we have the following theorem.

Theorem 2.1. *Every non-planar graph contains a strict regular odd ring.* ▮

We shall prove that any graph with a strict regular odd n-ring with $n \geqq 5$ must be non-planar. First we need one more definition and a lemma.

If C is a circuit in the graph G, then a *C-avoiding path* is a path P in G for which $(IP \cap VC) \cup (EP \cap EC) = \emptyset$. The following lemma concerning the existence of a C_i-avoiding path is proved in [2].

Lemma 2.2. *Let S be a strict ring and let $C_r, C_s, C_i \in S$, where $C_i \notin \{C_r, C_{r+1}, \ldots, C_s\}$ and $\{C_r, C_s\} \cap \{C_{i-1}, C_i, C_{i+1}\} = \emptyset$. Then for any $u \in VC_r$ and $v \in VC_s$, there exists a C_i-avoiding path joining u and v.* ▮

We are now ready to prove the following theorem.

Theorem 2.3. *Let S be a strict regular odd n-ring, with $n \geqq 5$, in a graph G. Then G is non-planar.*

Proof. If S is elegant, then the result is immediate by Theorem 1.1.

Suppose that S is not elegant. Let $S=(C_0, C_1, \ldots, C_{n-1})$, and suppose that every circuit of S is a directed circuit. Since S is not elegant, there exists i such that there are at least two $\bar{C}_i C_{i+1}$-chords, P_1, and P_2. Let o_1, o_2 be the origins of P_1, P_2, respectively, and let t_1, t_2 be their respective termini. Since C_{i+1} is a circuit, and P_2 is a $\bar{C}_i C_{i+1}$-chord other than P_1, we may certainly assume that $o_2 \in IC_i(t_1, o_1) \cup \{t_1\}$. (Otherwise some subpath of $C_{i+1}(t_1, o_2)$ must be a $\bar{C}_i C_{i+1}$-chord with origin in $VC_i(t_1, o_1)$.)

Since S is regular, $EP_1 \cap EC_{i+2} \neq \emptyset$. Since S is strict and $n \geqq 5$, we have $EP_1 - EC_{i+2} \neq \emptyset$, for otherwise $o_1, t_1 \in EC_i \cap EC_{i+2}$. Combining these results, we see that there exists $p_1 \in IP_1 \cap VC_{i+2}$. Similarly there exists $p_2 \in IP_2 \cap VC_{i+2}$.

I. Suppose that $o_2 \in IC_i(t_1, o_1)$ and $t_2 \in IC_i(o_1, t_1)$. By Lemma 2.2, there exists a C_i-avoiding path Q joining p_1 and p_2. Let Q' be a subpath of Q, of minimal length, joining a vertex of IP_1 to a vertex of IP_2. Then $C_i \cup P_1 \cup P_2 \cup Q'$ is a subdivision of $K_{3,3}$ (see Figure 1), so that G is non-planar.

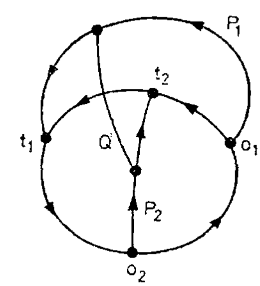

Fig. 1

II. Suppose that either $o_2 = t_1$ or $t_2 \in IC_i(t_1, o_1) \cup \{o_1\}$. If $t_2 \in IC_i(o_1, t_1)$, then define $R_1 = C_i(t_1, o_1)$ and $R_2 = C_i(t_2, t_1)$. If $t_2 \in VC_i(o_2, o_1)$, then define $R_1 = C_i(o_1, t_1)$ and $R_2 = C_i(o_2, t_2)$. If $o_2 \neq t_1$ and $t_2 \in VC_i(t_1, o_2)$, then define $R_1 = C_i(o_1, t_1)$ and $R_2 = C_i(t_2, o_2)$.

In each case, we note that $C_{i+1} \neq P_1 \cup R_1$ and $C_{i+1} \neq P_2 \cup R_2$, for $P_1 \cup R_1$ does not contain P_2 and $P_2 \cup R_2$ does not contain P_1. It follows that R_1 must contain a $\bar{C}_{i+1} C_i$-chord, A. Since S is regular, A contains an edge of EC_{i-1}. Since S is strict and $n \geqq 5$, A also contains an edge that is not in EC_{i-1} for otherwise both end-vertices of A would belong to $VC_{i+1} \cap VC_{i-1}$. It follows that IA contains a vertex of VC_{i-1}. Hence we may choose $r_1 \in IR_1 \cap VC_{i-1}$. Similarly we choose $r_2 \in IR_2 \cap$ $\cap VC_{i-1}$. There must therefore be a subpath B' of C_{i-1}, joining r_1 and r_2, which contains no vertex of VC_{i+1}, since S is strict. Let B be a subpath of B', of minimal length, joining a vertex of IR_1 to a vertex of $VC_i - VR_1$. Then B is a $\bar{C}_i C_{i-1}$-chord, and therefore contains edges of EC_{i-2}. By an argument used before, we may choose a vertex $v \in IB \cap VC_{i-2}$. By Lemma 2.2, there is a C_i-avoiding part D' in G joining v to p_1. Let D be a subpath of D', of minimal length, joining a vertex of IP_1 to a vertex of IB. Then $C_i \cup P_1 \cup B \cup D$ is a subdivision of $K_{3,3}$, and so G is non-planar. ∎

A (strict) regular 3-ring in a planar graph is exhibited in Figure 2. The three circuits of the ring are those with vertex sets $\{a, i, j, r, s, k, l, d, e, m, n, v, w, o, p, h\}$, $\{a, i, q, x, w, o, g, f, e, m, u, t, s, k, c, b\}$ and $\{b, j, r, q, x, p, h, g, f, n, v, u, t, l, d, c\}$ respectively. The arrows show these circuits are consistently orientable.

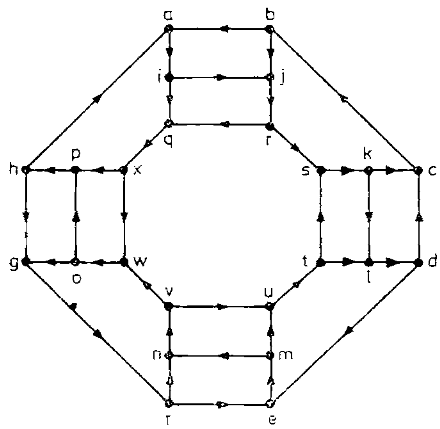

Fig. 2

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